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Approximating and Intersecting Surfaces from Points

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Abstract

Point sets become an increasingly popular shape representation. Most shape processing and rendering tasks require the approximation of a continuous surface from the point data. We present a surface approximation that is motivated by an efficient iterative ray intersection computation. On each point on a ray, a local normal direction is estimated as the direction of smallest weighted co-variances of the points. The normal direction is used to build a local polynomial approximation to the surface, which is then intersected with the ray. The distance to the polynomials essentially defines a distance field, whose zero-set is computed by repeated ray intersection. Requiring the distance field to be smooth leads to an intuitive and natural sampling criterion, namely, that normals derived from the weighted co-variances are well defined in a tubular neighborhood of the surface. For certain, well-chosen weight functions we can show that well-sampled surfaces lead to smooth distance fields with non-zero gradients and, thus, the surface is a continuously differentiable manifold. We detail spatial data structures and efficient algorithms to compute ray-surface intersections for fast ray casting and ray tracing of the surface.

Categories and Subject Descriptors (according to ACM CCS): G.1.2 [Numerical Analysis]: Approximation of surfaces and contours I.3.5 [Computer Graphics]: Curve, surface, solid, and object representations I.3.7 [Computer Graphics]: Ray Tracing

1. Introduction

Points samples without additional topological information gain popularity as a shape representation. On one hand, many shapes are nowadays created using sampling[36, 45], where the sampling process provides only partial connectivity information. On the other hand, points are a reasonable display primitive for shapes with high geometric or textural complexity relative to the rastered image[42, 46, 31, 53]. Since acquisition and rendering are point-based, it seems logical to stay within the framework of point-based shape representation also during the modelling stage of shapes[24, 52, 41].

Modelling or processing shapes, however, requires to interrogate the surface. For point representation this typically means to attach a continuous surface approximation to the points. Approximation of surfaces (and not just functions over a Euclidean domain) from irregularly spaced points is still a fairly young topic, where many approaches are rather practical and provide no guarantee that the reconstructed surface is, for example, continuous, manifold, or resembles the topology of the sampled surface. Interestingly, only few attempts have been made to give a criterion for sufficient sampling of a surface – a notable exception is the line of work initiated by Amenta and co-workers[4, 5, 7] Here we present a scheme for the approximation of *smooth* surfaces from irregularly sampled points that also allows formulating a sampling criterion, however, not yet as concise as Amenta's.

Our approach resembles a ray tracing technique[1] for Point Set Surfaces[35, 2, 3], a surface approximation that uses a non-linear projection operation to define the surface as the stationary points of the projection. It has been conjectured that the projection operation gives rise to a continuous manifold reconstruction. In an attempt to speed up the ray intersection computation, we have replaced the non-linear projection during ray intersection with a simpler method. We found that the surface that is implicitly defined by this operation has, in fact, comparable properties – only they are easier to prove. The requirements for the reconstruction being a continuous manifold lead to a natural and intuitive sampling criterion.

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After establishing some context by briefly discussing related work (Section 2), we first provide the theory using a slightly less general version of our surface definition (Section 3) and then explain the iterative procedure (Section 4) and spatial data structures (Section 5).

2. Related Work

We concentrate on work that is directly related to our approach, namely, the approximation of a surface from point samples and interrogating this surface by means of fast ray-surface intersections.

Most approximation techniques define the surface implicitly, either by defining a scalar function of space or by certain constructive means.

An interesting line of approximation algorithms define the surface as a subgraph of the Delaunay complex of the points[12, 19]. Many algorithms follow this spirit and differ mostly in how they identify triangles that belong to the surface. Crust[4, 5] (or power crust[7]) uses vertices of the Voronoi diagram (power diagram in case of power crust) as an approximation to the medial axis. The Delaunay triangulation including these vertices connects points on the surface either to the medial axis or to natural neighbors, which allows identifying surface triangles. The connection to the medial axis leads naturally to a minimal sampling density that is linear in the proximity of the surface to the medial axis. Sufficient sampling guarantees a reconstruction of the original topology. Cocone[6, 16, 17] has similar guarantees but eliminates the step of adding Voronoi vertices to the point set. This makes the Delaunay-complex significantly smaller and, thus, the reconstruction faster. Still, constructing the Delaunay complex of millions of points is costly and some algorithm rather use local triangulations of the points[11, 22]

Hoppe et al.[28] defines an implicit function that is interestingly in a sense dual to Delaunay-type reconstruction: For each point a normal direction is estimated from neighboring points and all normals are oriented consistently. The signed distance to the surface is defined as the normal component of the distance to the closest point. Thus, the surface consists of planes through the points bounded by the Voronoi cells of the points.

In many practical cases one has multiple point samples for the same region. A way to consolidate this information is to build a distance function in a volumetric grid by properly weighting the points[15, 51].

Defining the surface as a set of planar pieces results in C^0 approximations. To achieve smoother approximations one could build a smooth surface over the triangulation[27], blend the planar pieces using their implicit representation[13], or fit smoother functions. A global and smooth interpolant for scattered data can be constructed using radial basis functions (RBF). For surface approximation an implicit function is computed using extra points away from the surface[47, 49]. The computation traditionally involves the solution of a large linear system, however, is nowadays tackled using compactly supported functions[50], mulitpole expansions[8, 10, 9], thinning[20, 21, 29, 18], or hierarchical clustering[30, 39].

Another approach is to fit globally smooth functions locally[23], or to perform purely local fits[43, 33] and blend these local surface approximations together[38]. The moving least squares (MLS)[34] approximation takes this approach to the extreme by building a local fit for every point on the surface. Using MLS allows defining a projection operation that defines the surface implicitly as its stationary points[35]. The projection operation could be used for resampling the surface[2, 3]. Our surface definition results from simplifying a ray intersection procedure[1] for this type of surface. To suit uneven sampling one could use locally adaptive weight functions[40, 41], however, it seems difficult to adapt the weighting in a smooth way, which would be necessary to remain the manifold conjectures of the MLS approach.

For modelling and rendering the surface has to be interrogated. While for rendering one could use the the existing points[42, 46, 31, 53], modelling typically requires operations such as ray-surface intersection, for example, to specify points on the surface by clicking. For a variety of deformation and CSG operations the MLS projection operator could be used[41].

Computing ray-surface intersections for an implicit surface is conceptually simple: The ray is substituted in the implicit surface definition. Computing the intersection is, thus, equivalent to finding a root of a function in one unknown. To speed up the intersection computation one should exploit properties of the implicit function. A common way is to compute a (local) Lipschitz constant, which yields a conservative step width[32, 26]. Another approach is to use interval analysis[37].

Schaufler and Jensen define a ray-surface intersection for point sets directly, without an intermediate surface definition[48]. They collect points within a cylinder around the ray and compute a weighted average surface location. This is very fast, however, the geometry resulting from ray surface intersection depends on the particular rays used for intersecting the surface.

3. Foundations – Simple Surface Definition and Sampling Criterion

We assume that a set of points implicitly defines a smooth manifold surface. More specifically, let points $p_i \in \mathbb{R}^3, i \in \{1,\ldots,N\}$, be sampled from a surface S (possibly with a measurement noise). The general idea of our surface definition is inspired by MLS approximation – the surface is approximated by building local polynomial approximations everywhere in space and a point $\mathbf{s}$ in space belongs to the surface if its local polynomial approximation contains $\mathbf{s}$. For

reasons of clarity we first describe a slightly simplified version of the definition. We feel this makes the connection to the sampling criterion and the resulting properties easier to establish. The more general surface definition is given later together with an algorithm that computes ray surface intersections.

The main tools for the definition of the surface are weighted averages and weighted co-variances of the points in a tubular neighborhood around the surface. A weight function $\theta : \mathbb{R} \to \mathbb{R}$ specifies the influence of a point $\mathbf{p}_i$ using the euclidean distance, i.e. $\theta_i(\mathbf{x}) = \theta(\|\mathbf{p_i} - \mathbf{x}\|)$. Weight functions are assumed to be smooth, positive, and monotonically decreasing (have negative first derivative).

The weighted average of points at a location $\mathbf{s}$ in space is

$$\mathbf{a}(\mathbf{s}) = \frac{\sum_{i=0}^{N-1} \theta_i(\mathbf{s})\mathbf{p}_i}{\sum_{i=0}^{N-1} \theta_i(\mathbf{s})} \tag{1}$$

and the weighted co-variance at $\mathbf{s}$ in direction $\mathbf{n}$ describes how well a plane $\mathbf{n}(\mathbf{s}-\mathbf{x}) = 0$ fits the weighted points:

$$\sigma_n^2(\mathbf{s}) = \frac{\sum_{i=0}^{N-1} \theta_i(\mathbf{s}) \left(\mathbf{n}(\mathbf{s}-\mathbf{p}_i)\right)^2}{\sum_{i=0}^{N-1} \theta_i(\mathbf{s})} \tag{2}$$

Let $\sigma(\mathbf{s})$ be the vector of weighted co-variances along the directions of the canonical base

$$\sigma(\mathbf{s}) = \begin{pmatrix} \sigma_{(1,0,0)}(\mathbf{s}) \\ \sigma_{(0,1,0)}(\mathbf{s}) \\ \sigma_{(0,0,1)}(\mathbf{s}) \end{pmatrix} \tag{3}$$

then the major axes (i.e. directions of smallest and largest weighted co-variance at a point $\mathbf{s}$) are accessible as the eigenvectors of the bilinear form

$$\Sigma(\mathbf{s}) = \sigma(\mathbf{s})\sigma(\mathbf{s})^T \tag{4}$$

where an eigenvalue is the co-variance along the direction of the associated eigenvector.

Our computation and definition of the surface mainly depends on local frames, which are built from locally estimated normals.

Definition 1 The *normal direction* $\mathbf{n}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3$ (or *normal* for short) is defined as a direction of smallest weighted co-variance, i.e. $\min_n \sigma_n^2(\mathbf{x})$. If n is unique the normal is *well-defined*.

It is clear that the normal in $\mathbf{x}$ is given as the eigenvector of the co-variance matrix $\Sigma(\mathbf{x})$ corresponding to the smallest eigenvector. The normal is well-defined exactly if $\Sigma(\mathbf{x})$ has an eigenvalue that is strictly smaller than all other eigenvalues.

We define the surface implicitly based on normal directions and weighted averages. The implicit function $f : \mathbb{R}^3 \to \mathbb{R}$ describes the distance of a point $\mathbf{x}$ to the weighted average $\mathbf{a}(\mathbf{x})$ projected along the normal direction $\mathbf{n}(\mathbf{x})$:

$$f(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot (\mathbf{a}(\mathbf{x}) - \mathbf{x}) \tag{5}$$

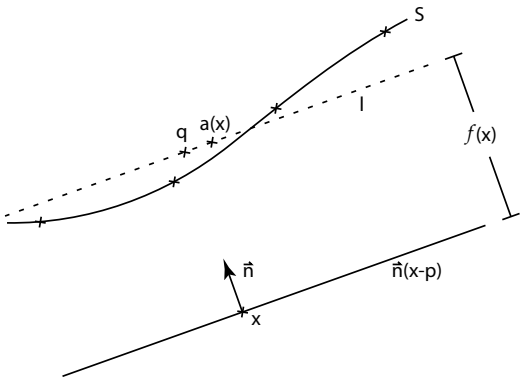

Figure 1: *The surface is defined implicitly as the zero set of a function $f(\mathbf{x})$. In each point $\mathbf{x}$ a local normal direction $\mathbf{n}(\mathbf{x})$ is estimated as the direction of minimal weighted co-variance. The implicit function $f(\mathbf{x})$ describes the distance of a weighted average $\mathbf{a}(\mathbf{x})$ of the points along normal direction.*

As always, the approximated surface is defined as the zero-set of the implicit function, i.e.

$$\mathcal{S} = \{\mathbf{x} : f(\mathbf{x}) = 0\} \tag{6}$$

We know from differential geometry that $\mathcal{S}$ is a smooth differentiable manifold if f is a smooth function with non-zero gradient at least in an ε-tubular neighborhood around the zero-set (this a generalization of the inverse function theorem in calculus[14, 25]). Requiring f to be smooth leads to a surprisingly simple and natural sampling criterion:

Definition 2 A surface S is *well-sampled* with points $\{\mathbf{p}_i\}$ and approximated by $\mathcal{S}$ if the normals are well-defined inside a neighborhood around S that encloses the zero-set of f.

We will show that this condition is sufficient for f being smooth for points $\mathbf{x}$ inside the tubular neighborhood: First note that f is a smooth function in $\mathbf{a}$ and $\mathbf{n}$. If all weight functions are smooth, the weighted average $\mathbf{a}$ and the weighted co-variance matrix are smooth functions in $\mathbf{x}$. Furthermore, eigenvalues are smooth functions in the matrix coefficients and eigenvectors are the solution of a linear system in the eigenvalues and the matrix. Since the normal direction is defined as the eigenvector corresponding to the smallest eigenvalue, $\mathbf{n}(\mathbf{x})$ is smooth in $\mathbf{x}$ as long as one eigenvector is always associated with the smallest eigenvalue. This has to be the case if the smallest eigenvalue is always strictly smaller than all other eigenvalues, i.e., if all normals are well-defined.

Note that it is no surprise (from a differential geometry point of view) that a well defined normal field defines a surface. The main point here is that the definition of normals

as directions of smallest weighted co-variance admits such a definition.

The topology of the approximated surface $\mathcal{S}$ depends on the particular choice of weight functions θ_i. It is clear that certain θ_i could lead to non-manifold approximations and that even if $\mathcal{S}$ is manifold it's not necessarily homeomorphic to S. On the other hand, weights could be so chosen that $\mathcal{S}$ is manifold and, with further restrictions, resembles the topology of S.

To give an intuition for this, we first consider an infinitesimal ball B around $\mathbf{s}$. Inside the ball, weights are constant and so are $\mathbf{a}(\mathbf{x})$ and $\mathbf{n}(\mathbf{x})$. Thus, for this region f describes a plane and if this plane passes the ball the approximation of $\mathcal{S}$ inside B is a disk. If, furthermore, the gradient of f inside such ball is non-zero, $\mathcal{S}$ is manifold. We show in the Appendix that using Gaussians as weight functions is sufficient for non-zero gradient at $f(\mathbf{x}) = 0$.

For homeomorphic reconstruction the support of the weights should be so chosen that they separate different sheets of the surface (note that sufficient sampling is a prerequisite for differentiation of sheets). As Gaussians have infinite support we have no practical means to construct theoretically correct weights yet, however, in practice Gaussians with appropriate radius perform quite nicely (as is demonstrated in Section 6).

4. Ray-surface intersections

The surface definition given in the previous section implies a technique to efficiently compute ray-surface intersections. The idea is to evaluate function f, as it provides a rough approximation of the distance field to $\mathcal{S}$ for $f(x) \neq 0$. For fixed $\mathbf{n}$ and $\mathbf{a}$ Equation 5 describes the planar fit

$$l(\mathbf{x}) = \mathbf{n}(\mathbf{s}) \cdot (\mathbf{a}(\mathbf{s}) - \mathbf{x}) \tag{7}$$

to S with respect to the location $\mathbf{s}$ (depicted by the dashed line in Figure 1). The smaller the distance $f(\mathbf{s})$ is, the better S is approximated in $\mathbf{q}$, which is the projection of x onto l along n. If $f(\mathbf{s}) = 0$, $\mathbf{q} = \mathbf{s}$ is a point on the surface $\mathcal{S}$.

The approximation l is used to converge to S along a ray r, using an iterative scheme similar to the ray tracing approach for MLS-surfaces[1]. Once an approximation is determined, the equation $r = r_o + s \cdot t_d$ is inserted in $l(x)$, and solving $l(\mathbf{r}) = 0$ provides t_d for the intersection of ray and planar surface approximation. The series of intersections $\{r_i\}$ approaches the surface $\mathcal{S}$. In theory, once $\mathbf{r}_i$ is close enough to the surface, the series $|f(\mathbf{r}_i)|, |f(\mathbf{r}_{i+1})|, |f(\mathbf{r}_{i+2})|, \ldots$ is strictly decreasing and an increase could be used to bail out off the iteration and start over. In practice, however, we have to accommodate imperfect weighting and use more tolerant iteration conditions: We require that the closest point $\mathbf{q}$ on the approximated surface is close to the current position on the ray $\mathbf{r}_i$ that has been used to compute $\mathbf{q}$ as well as close to the next intersection $\mathbf{r}_{i+1}$ of the ray and the plane $l = 0$ through $\mathbf{q}$. Specifically, a region of trust T around $\mathbf{q}$ has to contain $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$. If not, the iteration is terminated and no ray surface intersection in the proximity of $\mathbf{r}_0$ could be reported.

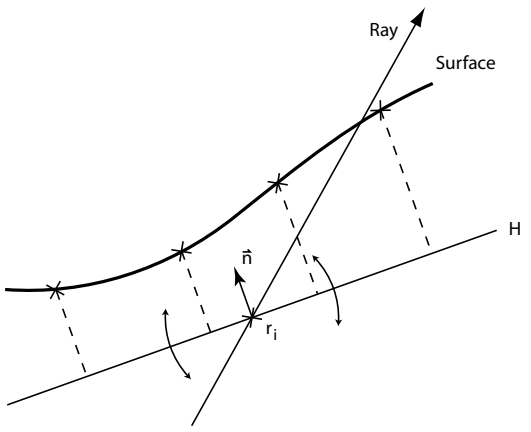

Figure 2: *In an intermediate approximation of the ray surface intersection $\mathbf{r}_i$ a local coordinate system H is established using the direction of minimal weighted co-variance as the normal* $\mathbf{n}$

How to obtain an initial point $\mathbf{r}_0$ is discussed later in Section 5. The region T depends on the weighting-function for $n(x)$ and $a(x)$, which affects the size of features in $\mathcal{S}$. How to make suitable choices for the weighting-function and T is also addressed in Section 5.

The procedure sketched above is easily generalized by taking the following point of view: In each point on $\mathbf{r}$ a local coordinate system is created using the approximation of a normal direction. In this coordinate system a weighted least square *constant* approximation to the surface is computed (the weighted average is a constant least squares approximation). This local approximation is intersected with the ray and the procedure is repeated. In this setting it seems quite natural to use higher order least squares approximations to the surface in the local coordinate system in an attempt to increase the approximation order of the scheme. In retrospect, we have used constant polynomials for the analysis because this allows explicitly solving the linear system that determines the polynomial.

The following three steps are iterated until the sequence $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \ldots$ converges to the ray-surface intersection or the procedure is terminated, focussing on the next region to be examined:

1. **Support plane:** The normal of a support plane H in $\mathbf{r}_i$ is determined by minimizing the weighted distances of the points $\mathbf{p}_j$ to H. The weights are computed from the distances of the $\mathbf{p}_j$ to $\mathbf{r}_i$ using a smooth, positive, monotone decreasing function θ (e.g. a gaussian $\theta(d) = e^{-\frac{d^2}{h^2}}$). This

weighted least squares problem is solved by minimizing

$$\sum_{j=1}^{N} \big(\langle \mathbf{n}, \mathbf{p}_j - \mathbf{r}_i \rangle\big)^2 \theta\big(\|\mathbf{p}_j - \mathbf{r}_i\|\big), \tag{8}$$

The minimization of equation 8 can be rewritten in bilinear form

$$\min_{\|\mathbf{n}\|=1} \mathbf{n}^T B \mathbf{n}, \tag{9}$$

where $B = \{b_{kl}\}$ is the matrix of weighted co-variances

$$b_{lk} = \sum_{j=1}^{N} \theta_j \big(p_{j_k} - r_{i_k}\big)\big(p_{j_l} - r_{i_l}\big).$$

The minimization problem in equation 9 is solved by computing the eigenvector of B with the smallest eigenvalue. The resulting H is approximately parallel to the Surface in the area around its nearest approach to r_i.

2. **Polynomial approximation:** The support plane in $\mathbf{r}_i$ is used to compute a local bivariate polynomial approximation a_i of $\mathcal{S}$. To determine the coefficients of a_i, again a weighted least squares problem is solved by minimizing the equation

$$\sum_{j=1}^{N} \big(a_i(x_j, y_j) - f_j\big)^2 \theta\big(\|\mathbf{p}_j - \mathbf{r}_i\|\big), \tag{10}$$

where (x_j, y_j) is the projection of $\mathbf{p}_j$ onto H in normal direction and $f_j = \langle \mathbf{n}, \mathbf{p}_j - \mathbf{r}_i \rangle$ is the height of $\mathbf{p}_j$ over H. Equation 10 can be minimized by calculating its gradient over the unknown coefficients of the polynomial. This leads to a system of linear equations, which is solved using standard numerical methods. The resulting polynomial is a local approximation of the surface $\mathcal{S}$. If $\mathbf{r}_i$ is sufficiently close to $\mathcal{S}$, a_i is expected be a good approximation to $\mathcal{S}$ around $\mathbf{r}_i$ and the intersection $\mathbf{r}_{i+1}$ of the ray with a_i could be used to converge to $\mathcal{S}$.

3. **Intersection:** If the ray intersects a_i, this point r_{i+1} serves as the starting point for the next iteration. If the ray misses a_i the iteration is terminated. Only ray-polynomial intersections within T are considered. An intersection is detected, when the constant part c of a_i is zero. In practice a c being smaller then a ε suffices to accept r_i (or the ray-intersection with a_i) as an adequate ray-surface intersection.

5. Spatial data structures

In this Section we describe how to represent a tubular neighborhood around the surface. This neighborhood is needed to make sure that the intersection procedure described earlier starts from a suitable point $\mathbf{r}_0$. Moreover, using simple primitives for the representation of the neighborhood significantly speeds-up ray surface intersection. In practice, the size of a tubular neighborhood around the surface that contains only well-defined normals is unknown a-priori. Our best choice is to construct a spatial region that has a certain maximum distance to the point set as we expect the distance of the points to the reconstructed surface to be bounded.

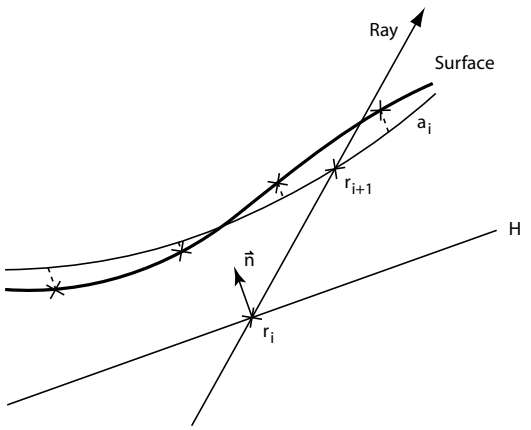

Figure 3: *The local coordinate system H is used to compute a local bivariate polynomial approximation to $\mathcal{S}$. This approximation is intersected with $\mathbf{r}$ to yield the next approximation to the ray surface intersection $\mathbf{r}_{i+1}$.*

Specifically, we construct a set of balls B_i of radius ρ_i around the points p_i. If the surface $\mathcal{S}$ is contained in the union of the balls the balls are a bounding volume of $\mathcal{S}$ that is easy to test for intersection. If a ray intersects $\mathcal{S}$, it also intersects at least one ball containing the intersection. Thus, an intersected ball indicates a potential ray surface intersection. The radius h has to be chosen, to ensure that

$$\mathcal{S} \subseteq \sum_{i=0}^{N} B_i. \tag{11}$$

Unfortunately, $\mathcal{S}$ is unknown a priori. The only a-priori knowledge are the points p_i, which are expected to be very close to the surface $\mathcal{S}$. Therefore, we choose conservative radii ρ_i, so that each B_i encloses the k-nearest neighbors of $\mathbf{p}_i$. In practice, we use $k = 6$.

The intersection of a ray with the set of balls can be computed efficiently. To quickly determine a subset of potentially intersected balls, the balls are arranged within an octree (see Figure 4), which is traversed along the ray using a parametric algorithm [44]. The current octree-voxel provides the candidate balls to be tested against the ray. Intersected balls are sorted along the ray to ensure that the first ray surface intersection is computed.

Each ray ball intersection is handled as follows: The center $\mathbf{p}_i$ of B_i is used as initial point $\mathbf{s}$ for the construction of a local coordinate system and polynomial approximation a_i. Using $\mathbf{p}_i$ for this approximation instead of the ray ball intersection has two reasons: First, $\mathbf{p}_i$ is expected to be close to $\mathcal{S}$ and should provide a reasonable approximation of the

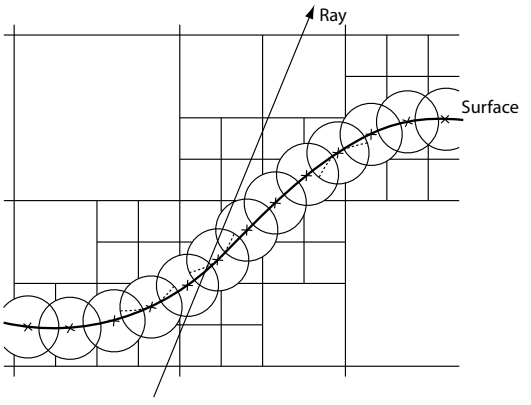

Figure 4: *The spatial data structures used to represent a tubular neighborhood around $\mathcal{S}$ is constructed as union of balls around the points. This region has the property that it contains all points with a certain maximum distance to the points and represents a best guess to the tubular neighborhood as points are expected to be close to the surface. For reducing the number of ray sphere intersections an octree is used.*

surface around $\mathbf{p}_i$. Second, the coordinate system and polynomial approximation within B_i are independent of the ray and can be stored for intersection with the next ray that intersects B_i. Intersecting the ray with a_i yields $\mathbf{r}_0$. Figure 4 illustrates this idea.

Then the procedure detailed in Section 4 is applied using B_i as the region of trust T. Thus, if the ray intersects the polynomial approximation inside B_i, the step is repeated until the desired accuracy is reached; otherwise no ray surface intersection is found within B_i and the next intersected ball along the ray is inspected.

Sometimes it is not important to determine the first intersection along a ray but only if there is any intersection with the object. A prominent example for rendering algorithms are shadow rays in ray tracing. Once a shadow-ray is obstructed by an opaque object, it is not necessary to determine at what position the ray first hits that object. Such a ray can be simply discarded from the illumination computations. This specific ray intersection query can be optimized by finding an intersection as quickly as possible anywhere on the ray. As spheres intersected close to their center are more likely to contain an intersection with the surface, they are sorted according to the distance d_i from the ray to the center c_i. The following equation determines the priority γ_i of a sphere considering different radii ρ_i.

$$\gamma_i = \frac{\rho_i - d_i}{\rho_i}. \tag{12}$$

6. Applications & Results

We have used the ray surface intersection algorithm to compute renderings by means of ray tracing. In practice, we use Gaussian weights, i.e. $\theta_i(\mathbf{x}) = \exp(\|\mathbf{x}-\mathbf{p}_i\|^2/h^2)$. The global parameter h allows specifying the locality of the approximation. Using smaller values for h results in a more local approximation, larger values could be used to smooth out small variations in the surface (e.g. noise). Since rendering is very fast, estimating useful values for h is done interactively. Note that for uneven sampling a localized Gaussian weighting has proven to be beneficial for MLS projection operation[41]. We have found our surface to exhibit less artifacts than the surface defined from the MLS projection so that we have not yet experimented with varying values for h.

To analyze the performance of the ray intersection algorithm we have computed several renderings of the Cyberware Rabit Model consisting of 67,038 points. Connectivity information available in the original data was discarded.

The effect of using different values for h is shown in Figure 5: the left rabbit results from using $h = 0.00375d$ and the right using $h = 0.017d$, where d is the object's diameter; as expected, larger values for h yield a smoother surface. Using smaller values for h leads to undefined normals, because the sampling is not dense enough compared to the weight function. This illustrates the effect of reconstructing from undersampled surfaces.

The following timings have been acquired using $h = 0.004d$ and an image raster of 200x400 pixel on a P4 with 2GHz: In 10.3 seconds ray surface intersections for 42,463 of the 80,000 rays were computed. In total the distance function was evaluated 238,715 times (each evaluation requires estimating a normal and computing the polynomial approximation). Roughly half of the evaluations lead to an intersection of the surface, the other half leads to bailing out of the iteration. If the center polynomials are stored and reused only 83,922 evaluations have to be calculated, where we compute and store the center polynomials on the fly.

To estimate the overhead of computing the pixel intensities and intersecting the rays with the spatial data structures we have substituted the ray surface intersection procedure with intersecting precomputed polynomials in the sphere centers. This simplification needs 1.3 seconds to ray trace the same scene. Apparently, most of the time is spent calculating and intersecting polynomial approximations.

An average 2.91 iterations were sufficient to satisfy a predefined precision of $p = 10^{-3}h$, which seems sufficient as features are expected to be larger than h. Table 1 shows the average number of iterations until convergence to ray surface intersection and the time to render the whole image relative to the required precision. Increasing the precision by an order of magnitude results in about 1.5 times iterations in average. The maximum precision that could be achieved is about $10^{-10}h$. Increased the required precision further leads to a

Figure 5: *Cyberware's Rabbit, consisting of 67,038 points, ray traced to images with 200x400 pixels using different Gaussian weighting functions. If the radius of the weights becomes too small, the surface is effectively undersampled w.r.t. to the weighting, thus, illustrating the effect of insufficient sampling density. Values for the parameter h relative to the objects diameter from left to right: 1.7%, 0.375%, 0.22%, 0.19%, 0.17%.*

Precision (h)	10^{-1}	10^{-3}	10^{-7}	10^{-10}	10^{-11}
Avg. Iter.	1.99	2.91	4.98	6.56	10.4
Time (sec)	7.9	11.5	18.9	24.6	42.7

Table 1: *Average number of iterations until convergence to a ray surface intersection and time needed to render an image at resolution of 200x400 pixels relative to the required precision.*

numerical breakdown of the procedure, possibly due to the eigenvector computation. This explains the superlinear number of iterations and computation time in the last column of the table.

7. Conclusions

We have presented a surface approximation technique that is based on an iterative ray-surface intersection algorithm. The definition of the surface allows deriving an intuitive criterion for sufficient sampling given a weighting function for the points. As the surface is defined by the ray intersection algorithm, ray tracing is a natural way to render the surface. Compared to ray tracing point set surfaces[1] our new approach is two orders of magnitude faster. It is comparable in speed to Schaufler & Jensen's approach[48], however, using a solid surface definition.

We admit that our formulation of the sampling criterion has several loose ends and that we are far from having a solid theory, nevertheless, we felt the results are useful and interesting. In particular, a qualitative and quantitative comparisons between the sampled surface S, the reconstruction we propose $\mathcal{S}$, and reconstructions with other methods are missing. From a practical perspective, important next steps are the definition of weights from a given smooth surface and the minimal extent of the tubular neighborhood. This would make the sampling criterion sufficient, yet still not very practical: One could only decide that a surface is not well-sampled by finding a point inside the neighborhood with undefined normal, which is very unlikely. Rather, we need conditions that necessarily lead to sufficient sampling (possibly accepting some oversampling).

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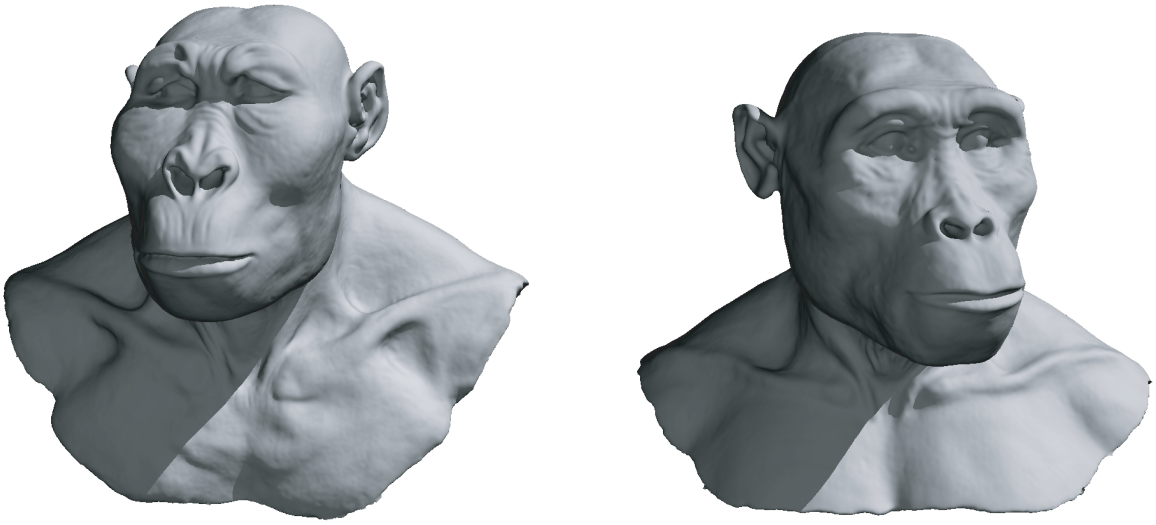

Figure 6: *Ray tracings of raw point data acquired with a structured light scanner. Both models have roughly 200K points and rendering an image with 906x868 pixel requires about 3 minutes on a P4 2GHz.*

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Appendix A: A sufficient condition for manifold reconstruction

To guarantee that $f(\mathbf{x}) = 0$ is manifold (assuming that the surface is well-sampled and, thus, f is smooth) we have to show that $\nabla f(\mathbf{x}) \neq 0$ for $f(\mathbf{x}) = 0$. We differentiate along normal direction in $\mathbf{x}$ to reduce the problem to one dimension. Note that this is possible because the surface is well-sampled, which implies that normals are well-defined. We can re-write f in a coordinate system with the origin on the surface along the dimension in normal direction as

$$f(\xi) = n(\xi)(a(\xi) - \xi) = \frac{\sum \theta_i(\xi)(\xi - \phi_i)^2}{\sum \theta_i(\xi)} \left(\frac{\sum \theta_i(\xi)\phi_i}{\sum \theta_i(\xi)} - \xi \right) \tag{13}$$

where ϕ_i denotes the distance of $\mathbf{p}_i$ along normal direction. Since we have defined differentiation w.r.t. ξ in direction of the normal n, $f(\xi) = 0$ implies $a(\xi) = 0$ so that we have to show only $a'(\xi)$ to be non-zero:

$$a'(\xi) = \frac{\sum \theta_i'(\xi)\phi_i \sum \theta_i(\xi) - \sum \theta_i(\xi)\phi_i \sum \theta_i'(\xi)}{\left(\sum \theta_i(\xi)\right)^2} - 1 \neq 0 \tag{14}$$

A particular simple way to satisfy this inequality is to use strictly positive weight functions of the form $\theta' = c\theta$ (i.e. exponential functions) because then the nominator is equal zero and the denominator is strictly positive. However, also Gaussian weight functions

$$\theta_i(h) = e^{|\xi - \phi_i|/h^2} \tag{15}$$

satisfy Eq. 14 for almost all h: The main observation is that

$$\sum \theta_i'(h)\phi_i \sum \theta_i(h) - \sum \theta_i(h)\phi_i \sum \theta_i'(h) - \sum \theta_i(h) \sum \theta_i(h) \tag{16}$$

is a smooth function in h, whose gradient cannot vanish everywhere because the Gaussians have strictly positive derivatives for positive values of h. Thus, Eq. 16 is smoothly varying with h and has only finitely many zeroes. All h that equate to non-zero values are sufficient for a local manifold reconstruction. Note that this is not sufficient for the existence of a global h that guarantees manifoldness everywhere on the surface, on the other hand, it is very unlikely in practice that the set of forbidden h-values all over the zero-set of f is dense in $\mathbb{R}$. A better characterization of admissible values for h is, nevertheless, desirable.